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Group actions on order trees

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Abstract

We begin an investigation of group actions on order trees. We develop some basic definitions and properties. When G is the fundamental group of a non-Haken Seifert fibered space, we completely describe all minimal order tree actions of G by showing that any nontrivial minimal action is necessarily dual to a foliation transverse to the Seifert fibering of M.

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1. Introduction

Much is now known about isometric group actions on $\mathbb{R}$-trees. There are several excellent survey articles; see, for example, [27,32,35,36]. $\mathbb{R}$-trees naturally arise as spaces of leaves of measured laminations in 3-manifolds. In this situation, the fundamental group of the 3-manifold acts by isometries on the space of leaves. On the other hand, when G is the fundamental group of a 3-manifold M, then Morgan and Shalen [30] showed that any nontrivial action of G on an $\mathbb{R}$-tree gives rise to an incompressible measured lamination in M and that M therefore contains an incompressible surface. However, in a certain sense [9,13,14,22,37,38], "most" 3-manifolds do not contain incompressible surfaces. Therefore, "most" 3-manifold groups do not act nontrivially on $\mathbb{R}$-trees. In attempting to understand these 3-manifolds which do not contain incompressible surfaces, one turns to a more commonly found object, the *essential lamination* [20]. In [20], Gabai and Oertel introduce the notion of *order tree* to describe the space of leaves of an essential lamination.

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A compact manifold which contains an essential lamination but not an incompressible surface has fundamental group which acts nontrivially and minimally on an order tree (see Section 8) although not on an $\mathbb{R}$-tree.

Very little is known about group actions on order trees. Gabai has asked (Problem 4.3, [16]) whether every nontrivial 3-manifold group action on an order tree is in some sense dual to an essential lamination. The question remains open even in the case where the space of leaves is $\mathbb{R}$.

In this paper, we begin an investigation of group actions on order trees. We develop some basic definitions and properties. We follow tradition by first examining non-Haken Seifert fibered spaces, those Seifert fibered spaces which do not contain an incompressible surface. When G is the fundamental group of a non-Haken Seifert fibered space M, we completely describe all minimal order tree actions of G by showing that any nontrivial minimal order tree action is necessarily dual to a foliation transverse to the Seifert fibering of M. Finally, we note that this gives an alternate proof of a result due to Brittenham [5] and Claus [8] demonstrating the existence of nonlaminar 3-manifolds with infinite fundamental group.

In Section 2 we record some of the background on $\mathbb{R}$-trees, order trees, and essential laminations. In Section 3 we develop some basic properties of paths in order trees. In Section 4 we adapt definitions of minimal and trivial group actions on trees to the order tree case. In Section 5 we establish facts about general group actions on order trees. Section 6 is devoted to lemmas needed in the proof of Theorem 7.2. In Section 7, we show that if the fundamental group of an non-Haken Seifert-fibered space acts nontrivially on an order tree, then that order tree must essentially be $\mathbb{R}$. In Section 8 we show that when λ is an essential lamination, then the natural action of $\pi_1(M)$ on the space of leaves of λ is nontrivial and minimal.

2. Preliminaries and definitions

Let M be a 3-manifold. If M contains an incompressible measured lamination $\mathcal{L}$, we can lift $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ in the universal cover $\widetilde{M}$, and the measure on $\mathcal{L}$ induces a metric on the space obtained by identifying each nonboundary leaf in the lift of $\mathcal{L}$ to a point, and each connected component of $\widetilde{M} \setminus \widetilde{\mathcal{L}}$ to a point. This metric space T (called the *space of leaves* of $\mathcal{L}$) turns out to be an $\mathbb{R}$-tree [28], and has a natural action by $\pi_1(M)$. Moreover, Morgan and Shalen [30] have shown that given a nontrivial $\pi_1(M)$ action on an $\mathbb{R}$-tree T, one can set up a $\pi_1(M)$-equivariant map from $\widetilde{M}$ to T which induces an incompressible measured lamination $\mathcal{L}$ in M called a *pullback lamination* of the tree T. So $\mathcal{L}$ is a pullback lamination of its space of leaves T. Hence the notions of 3-manifold group actions on $\mathbb{R}$-trees and incompressible measured laminations in 3-manifolds are dual.

Now if M contains an essential lamination λ, not measured, one may still form the space of leaves. It no longer inherits a metric structure, but it is still a tree with linearly ordered segments. Adopting the viewpoint of [18], the space of leaves is defined as follows. (For basic definitions and background on essential laminations, we refer the reader to [20, 16,17].) If necessary, first extend λ to an essential lamination without isolated leaves by replacing each isolated leaf L with an I-bundle regular neighbourhood (Operation 2.1.1, [15]). Let $\tilde{\lambda}$ denote the lift of the resulting lamination to $\widetilde{M}$. Then $\sim$ is an equivalence relation on $\widetilde{M}$, where $x \sim y$ if and only if either x and y lie on a common leaf of $\tilde{\lambda}$ or x and y lie in the closure of a common complementary region of $\tilde{\lambda}$. The quotient space

$$T(\lambda) = \widetilde{M}/\sim$$

is called the *space of leaves of* λ. The action of $\pi_1(M)$ on $\widetilde{M}$ by covering translations induces an action of $\pi_1(M)$ on $T(\lambda)$, which we call the *standard action* of $\pi_1(M)$ on $T(\lambda)$.

In [20], Gabai and Oertel showed that $T(\lambda)$ is an *order tree*. An order tree T is a set T together with a collection of linearly ordered subsets called *segments*. If σ is a segment then $-\sigma$ denotes the same subset with reverse order. The segments satisfy:

- Each segment σ has distinct least and greatest elements, which we will denote by $i(\sigma)$ and $f(\sigma)$, respectively. (We also write $\sigma = [i(\sigma), f(\sigma)]$.)
- If σ is a segment, so is $-\sigma$.
- A closed subinterval of a segment is a segment if it has more than one element.
- Any two elements of T can be joined by a sequence $\sigma_1, \ldots, \sigma_k$ of segments such that $i(\sigma_j) = f(\sigma_{j+1})$ for all j.
- Given a cyclic word $\sigma_0\sigma_1 \ldots \sigma_{k-1}$ (where $i(\sigma_j) = f(\sigma_{j+1})$ for all j, $0 \leqslant j \leqslant k-2$, and cyclic means $f(\sigma_{k-1}) = i(\sigma_0)$), there is a subdivision of the σ's $\rho_0 \ldots \rho_{n-1}$ so that when adjacent pairs $(\rho)(-\rho)$ are cancelled, we have the trivial word.
- If $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2$ is a segment.

In [18], Gabai and Kazez show that $T(\lambda)$ is actually an $\mathbb{R}$-order tree; i.e., an order tree satisfying also:

- Each segment is order isomorphic to a closed interval in $\mathbb{R}$.
- $T(\lambda)$ is a countable union of segments.

Remark (*Our definition of order tree*). For simplicity of exposition, in this paper we will take the definition of order tree to be a set T together with a set of segments satisfying the first six axioms above, and in addition the first of the two $\mathbb{R}$-order tree axioms, namely that segments are order isomorphic to closed intervals in $\mathbb{R}$.

One special case of an order tree (or $\mathbb{R}$-order tree) is that T is an $\mathbb{R}$-tree. However, when viewing such a T as an order tree we forget any natural metric structure, for the actions considered are usually not isometries but instead are order tree isomorphisms (i.e., bijections which preserve the order tree structure). For example, Theorem 7.2 reveals that if the fundamental group of a non-Haken Seifert fibered space acts nontrivially and minimally on an order tree T, then necessarily $T = \mathbb{R}$. This action cannot be by isometries since M contains no incompressible surface. Hence, this action also cannot satisfy Bestvina's nonnesting property [25].

Remark (*Definition of the space of leaves*). Segments are images in $T(\lambda)$ of closed efficient arcs (i.e., arcs intersecting each leaf of $\tilde{\lambda}$ at most once) in $\widetilde{M}$. Because it leads to simplifications of statements and differs from the original definition only when λ has isolated leaves, we have chosen the definition of space of leaves which guarantees that segments are order isomorphic to closed intervals of $\mathbb{R}$. In the original definition of space of leaves T' [20], segments are again images of efficient arcs in $\widetilde{M}$, but the elements of T' are the non-boundary leaves of $\tilde{\lambda}$ and the components of $\widetilde{M} \setminus \tilde{\lambda}$. When λ has no isolated leaves, this definition agrees with the one given by $\sim$. Otherwise, segments are no longer necessarily order isomorphic to closed intervals in $\mathbb{R}$. For example, consider the case that $\widetilde{M} = \mathbb{R}^3$ and $\tilde{\lambda}$ is the essential lamination of $\mathbb{R}^3$ given by the parallel planes $(\mathbb{Z} + \frac{1}{2}) \times \mathbb{R}^2$. Then T' can be represented by the points $\mathbb{Z} + \frac{1}{2}$, and segments of T' are then the finite sets $[x, y] \cap (\mathbb{Z} + \frac{1}{2})$.

Remark (*Two basic facts*). When a closed 3-manifold M contains an essential lamination λ, its universal cover is necessarily $\mathbb{R}^3$ (Theorem 6.1, [20]). Furthermore, $(\widetilde{M}, \tilde{\lambda}) = (\mathbb{R}^3, \tilde{\lambda})$ is homeomorphic to $(\mathbb{R}^2, \tilde{\lambda}_0) \times \mathbb{R}$, where $\tilde{\lambda}_0$ is an essential lamination in $\mathbb{R}^2$ (Theorem 4.6, [19]).

3. Structure of order trees

In this section we develop some basic properties of order trees. We follow the development of properties of $\mathbb{R}$-trees given in [1,10]. We begin with some definitions and notation.

Definition 3.1. Let T be an order tree. A *path* from $x \in T$ to $y \in T$ is a sequence of segments $\sigma_1 \cdots \sigma_n$ with $f(\sigma_i) = i(\sigma_{i+1})$ for $1 \leqslant i < n$ and $i(\sigma_1) = x$ and $f(\sigma_n) = y$.

Definition 3.2. A *standard geodesic* from x to y is a path $\sigma_1 \cdots \sigma_n$ from x to y satisfying:

- $\sigma_i \cap \sigma_j = \emptyset$ if $|i - j| > 1$.
- $\forall j$, either $\sigma_j \cap \sigma_{j+1} = i(\sigma_{j+1}) = f(\sigma_j)$ or $\sigma_j \cap \sigma_{j+1} = (i(\sigma_j), f(\sigma_j)] = (f(\sigma_{j+1}), i(\sigma_{j+1})]$.

Let $S = \{(\sigma, \tau) \mid (i(\sigma), f(\sigma)] = (f(\tau), i(\tau)] \text{ and } i(\sigma) \neq f(\tau)\}$, where σ and τ are segments. We define a relation on S as follows: let $([x, z], [z, y]) \equiv ([x, z'], [z', y])$ if $\exists r \in (x, z] \cap (x, z']$ so that

$$[x, z] = [x, r] \cup [r, z], \qquad [z, y] = [z, r] \cup [r, y],$$

$$[x, z'] = [x, r] \cup [r, z'], \qquad [z', y] = [z', r] \cup [r, y]$$

(where the segments $[r, z]$ and/or $[r, z']$ are understood to be empty if $r = z$ or $r = z'$). Then $\equiv$ is an equivalence relation.

Definition 3.3. A *cusp* is an equivalence class of pairs of segments in S under the above equivalence relation $\equiv$.

Notation. Note that a cusp represented by a pair $([x, z], [z, y]) \in S$ is determined by the pair of points x and y, for by axiom 5 in the definition of order tree, any other pair in S

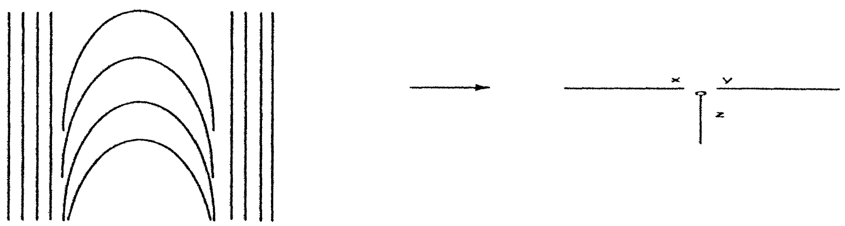

Fig. 1. “Non-Hausdorff branching”.

of the form $([x, z'], [z', y])$ must be in the same equivalence class. Hence we will use the symbol $[x, y]^c$ to denote the cusp. In this situation, we refer to the points x and y as *cusp points*.

Remark. When $T = T(\lambda)$ for some essential lamination $\lambda \subset M$, cusp points correspond exactly to leaves of $\tilde{\lambda}$ which cannot be separated by saturated open sets in $\widetilde{M}$. For example, let μ be the foliation of $[0, 1] \times \mathbb{R}$ given in Fig. 1, and choose $\lambda \subset S^1 \times S^1 \times S^1$ such that $\mu \times \mathbb{R} \subset \tilde{\lambda}$. Referring to the above notation, $x = i(\sigma_j)$, $y = f(\sigma_{j+1})$ and $z = f(\sigma_j) = i(\sigma_{j+1})$.

Then we have the following existence theorem.

Theorem 3.4. *Let T be an order tree. Given x and $y \in T$, $\exists$ a standard geodesic from x to y.*

Proof. Let $\sigma_1 \cdots \sigma_n$ be a path from x to y, which exists by the definition of order tree. Suppose that this path does not satisfy the first condition. Then $\exists p \in \sigma_i \cap \sigma_j$ where $j - i > 1$. Subdivide both σ_i and σ_j into pairs of subsegments at p, say $\sigma_i^1 \sigma_i^2$ and $\sigma_j^1 \sigma_j^2$. Deleting the subpath $\sigma_i^2 \cdots \sigma_j^1$ results in a new path $\sigma_1 \cdots \sigma_i^1 \sigma_j^2 \cdots \sigma_n$ which has at most $n - 1$ segments. After at most a finite number of such moves, the resulting path from x to y has the property that only adjacent segments have non-empty intersections.

Now consider a pair of adjacent segments $\sigma_j \sigma_{j+1} = [m, p][p, q]$. Given $r_0 \in \sigma_j \cap \sigma_{j+1}$ with $r_0 \neq p$, since $r_0 \in \sigma_j$ and $p \in \sigma_j$, $[r_0, p] \subseteq \sigma_j$. Similarly, $[p, r_0] \subseteq \sigma_{j+1}$. Therefore, $\sigma_j \cap \sigma_{j+1}$ is the union of all intervals of the form $[r_0, p]$ where $r_0 \in \sigma_j \cap \sigma_{j+1}$ and $r_0 \neq p$. Hence $\sigma_j \cap \sigma_{j+1}$ either has the form $[r, p]$ or $(r, p]$ for some $r \in T$.

If we are in the first case, a closed intersection, and $r \neq p$, subdivide and cancel to obtain $[m, r][r, q] = \sigma'_j \sigma'_{j+1}$, where now $\sigma'_j \cap \sigma'_{j+1} = r = f(\sigma'_j)$. (In the cases that j is the first segment in the path or $j + 1$ is the last segment in the path, then it is possible that $r = m$ or $r = q$ or both, and then σ'_j or σ'_{j+1} or both will be empty.) After performing this move at every closed intersection of adjacent segments, the resulting path still satisfies the first condition and now has the property that if two adjacent segments have closed intersection, they intersect only at the endpoint.

We may have some pairs $\sigma_j \sigma_{j+1} = [m, p][p, q]$ with half open intersection. If the intersection is $[p, p') = [p, q')$ we subdivide into $[m, p'][p', p][p, q'][q', q]$. Because the first condition is satisfied by the path, we may perform this independently at each open intersection, forming a path satisfying both conditions in the definition of a standard geodesic. □

Remark. In particular, if a path satisfies the first condition in the definition of a standard geodesic then no three segments in the path have a nonempty intersection. Hence each segment in a standard geodesic is a part of a representative of at most one cusp, or in other words it is never the case that (σ_j, σ_{j+1}) and $(\sigma_{j+1}, \sigma_{j+2})$ are both representatives of cusps.

Standard geodesic paths are not unique, but the lack of uniqueness as a set of points is all due to the lack of uniqueness in the representation of cusps as two segments. We make this more precise with the following definition.

Definition 3.5. Let γ be a standard geodesic from x to y. Define $GS_{(x,y)}$, the *geodesic spine* of γ, to be the union of all segments of γ which do not, together with an adjacent segment, give a representative of a cusp, together with all of the cusp points of γ.

Although $GS_{(x,y)}$ is defined as a set, it has a natural linear order inherited from the geodesic γ. We will sometimes abuse language and call the geodesic spine a path, even though it has gaps between cusp points. When a pair of segments representing a cusp $[p_1, p_2]^c$ is on γ, it is only p_1 and p_2 which are on $GS_{(x,y)}$. However, we will again sometimes abuse language and say that $[p_1, p_2]^c$ is on $GS_{(x,y)}$ to stress the fact that p_1 and p_2 are cusp points.

As the notation suggests, the geodesic spine of γ depends only on the endpoints of the geodesic. It is independent of the particular choice of standard geodesic γ. The uniqueness of the geodesic spine of a standard geodesic between x and y will follow from the following theorem.

Theorem 3.6. *The geodesic spine of a standard geodesic from x to y is the intersection of all paths from x to y.*

Proof. Let ρ be any path from x to y, and let γ be a standard geodesic from x to y. Subdividing the segments in ρ and γ if necessary, and cancelling pairs of segments of the form $(\sigma)(-\sigma)$, the path $\rho\gamma^{-1}$ cancels completely. Notice that the only way in which the segments σ and $-\sigma$ can both appear in a subdivision of γ is if a pair of segments of γ which represent a cusp contain σ as subsegments. Cancellations of this type never remove subsegments of segments in γ which are not part of cusp representatives, nor do they ever remove the cusp points in a cusp representative. So in order for the points in the geodesic spine of γ to be cancelled, they must all appear in ρ, and hence the geodesic spine of γ is a subset of ρ. Therefore, the geodesic spine of γ is contained in the intersection of all paths from x to y.

On the other hand, if a point p in the intersection of all paths from x to y, then $p \in \gamma$. Suppose p is in a cusp representative but is not a cusp point, say $p \in [m, r][r, q]$ where

$([m,r],[r,q])$ represents $[m,q]^c$ and $p \neq m,q$. Then choose $x \in (m,p) = (q,p)$, and subdivide at x and cancel. The new path, which is still a standard geodesic, contains $[m,x][x,q]$ and p no longer appears. Hence p is either a cusp point of γ, or p is in some segment which is not part of a cusp representative, so p is in the geodesic spine of γ. Therefore the intersection of all paths from x to y is contained in the geodesic spine of γ. □

We also define, for a pair of cusp points, the set of points which "lie along the cusp" as follows:

Definition 3.7. Given a pair of cusp points x and y, define the corresponding *cusp tree* $T[x,y]^c$ to be the set of all points p for which there exist $q \in T$ and γ, a standard geodesic from q to p, satisfying:

- $[x,q]\gamma$ is a standard geodesic from x to p.
- $[y,q]\gamma$ is a standard geodesic from y to p.
- $([x,q],[q,y])$ is a representative of $[x,y]^c$.
- $\gamma \cap [x,q] = \gamma \cap [y,q] = \{q\}$.

Given a cusp tree $T[x,y]^c$, we define its set of *limit cusp points* to be

$$\{x\} \cup \left\{z \mid T[x,y]^c = T[x,z]^c\right\} = \{y\} \cup \left\{z \mid T[x,y]^c = T[y,z]^c\right\}.$$

It is easy to see that there is a path between any two points in $T[x,y]^c$ which lies entirely within $T[x,y]^c$, and hence $T[x,y]^c$ is a sub-order tree of T.

Definition 3.8. Consider the set of all half open segments, that is sets of the form $[p,q)$ where $[p,q]$ is a segment of T. Consider the relation defined by $[p,q) \equiv [p',q')$ if $\exists r \in [p,q) \cap [p',q')$ with $[r,q) = [r,q')$. This is an equivalence relation, and we define a *ray* to be an equivalence class of half open segments under this relation.

It is clear from the definition of a cusp that all segments appearing in any representative of the cusp are all representatives of the same ray. Furthermore, elementary arguments show that if $[p,q)$ represents the same ray as any segment which is part of a pair representing $[x,y]^c$, then in fact $p \in T[x,y]^c$ and $[p,q) = [p,x) = [p,y)$. We call this ray the *distinguished ray* of the cusp tree.

Now that we understand paths between pairs of points in order trees, we move on to triples of points (compare to the Y-proposition in [1]).

Definition 3.9 ($X_{(x,y,z)}$ and $Y_{(x,y,z)}$). Let T be an order tree, and let $x,y,z \in T$. Set

$$\begin{aligned} X_{(x,y,z)} &= GS_{(x,y)} \cap GS_{(y,z)} \\ &= \{p \mid p \in \text{all standard geodesics from } x \text{ to } y \text{ and from } y \text{ to } z\}. \end{aligned}$$

Notice that $y \in X_{(x,y,z)}$ and that $X_{(x,y,z)}$ itself has the form of a geodesic spine except possibly at the end away from y. That is, it consists of a sequence of segments and cusp

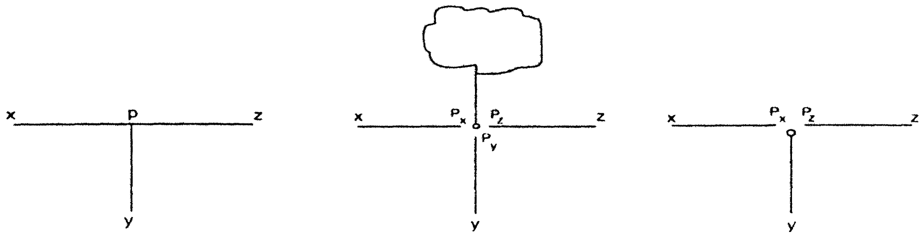

Fig. 2. Possibilities for Y.

pairs, beginning with one whose initial point is y, and ending in either an ordinary segment, a cusp pair, or an open segment. In either of the first two cases,

$$X_{(x,y,z)} = GS_{(y,p)},$$

for some $p \in GS_{(x,y)} \cap GS_{(y,z)}$, and we define $Y_{(x,y,z)} = p$. In the third case,

$$X_{(x,y,z)} = GS_{(y,p)} \setminus \{p\} = GS_{(y,q)} \setminus \{q\},$$

for some $p \in GS_{(x,y)}$, $q \in GS_{(y,z)}$, and we define $Y_{(x,y,z)} = [p, q]^c$.

There are very few ways a Y can be configured. They are listed in the following theorem and in Fig. 2 (where line segments are used to represent geodesic spines).

Theorem 3.10. *Let x, y, z be points in an order tree T. Then up to renaming the three points, one of the following three Y-configurations occurs.*

- $Y_{(x,y,z)} = Y_{(x,z,y)} = Y_{(y,x,z)} = p$.
- $Y_{(x,y,z)} = p_y$, $Y_{(y,x,z)} = p_x$, *and* $Y_{(x,z,y)} = p_z$, *where* p_x, p_y, *and* p_z *are distinct with common cusp tree* $T[p_x, p_y]^c = T[p_y, p_z]^c = T[p_x, p_z]^c$.
- $Y_{(x,y,z)} = [p_x, p_z]^c$, $Y_{(y,x,z)} = p_x$, *and* $Y_{(x,z,y)} = p_z$.

Proof. Consider first the case that one of the three Y's is a cusp. Without loss of generality, say $Y_{(x,y,z)} = [p_x, p_z]^c$. Then

$$X_{(x,z,y)} = GS_{(x,z)} \cap GS_{(y,z)} = GS_{(p_z,z)}$$

and hence $Y_{(x,z,y)} = p_z$. Similarly, $Y_{(y,x,z)} = p_x$. So we have the third possibility.

So we may suppose $Y_{(x,y,z)} = p_y$, $Y_{(y,x,z)} = p_x$, and $Y_{(x,z,y)} = p_z$. We show that the intersection

$$GS_{(x,y)} \cap GS_{(y,z)} \cap GS_{(x,z)}$$

contains at most one point. Suppose, by way of contradiction, that distinct points $s, t \in GS_{(x,y)} \cap GS_{(y,z)} \cap GS_{(x,z)}$. Then, after possibly relabeling s and t,

$$GS_{(x,y)} = GS_{(x,s)} \cup GS_{(s,t)} \cup GS_{(t,y)}$$

and

$$GS_{(x,z)} = GS_{(x,s)} \cup GS_{(s,t)} \cup GS_{(t,z)}.$$

But then $s \notin GS_{(y,z)} = GS_{(y,t)} \cup GS_{(t,z)}$, a contradiction.

Now consider first the case that

$$GS_{(x,y)} \cap GS_{(y,z)} \cap GS_{(x,z)} = \{s\}$$

is nonempty. By definition, $GS_{(y,p_y)} \cap GS_{(z,p_z)} = GS_{(x,y)} \cap GS_{(y,z)} \cap GS_{(x,z)}$, and so $GS_{(y,p_y)} \cap GS_{(z,p_z)} = \{s\}$. But this is possible only if $p_y = p_z = s$. By symmetry, it follows that $p_x = p_y = p_z$ and we have the first possibility.

Similarly, if

$$GS_{(x,y)} \cap GS_{(y,z)} \cap GS_{(x,z)} = \emptyset,$$

it follows that p_x, p_y, and p_z are pairwise distinct. Notice that $p_x, p_y \in GS_{(x,y)}$. Therefore, since $GS_{(x,p_x)} \cap GS_{(y,p_y)} = \emptyset$,

$$GS_{(x,y)} = GS_{(x,p_x)} \cup GS_{(p_x,p_y)} \cup GS_{(p_y,y)}.$$

By symmetry,

$$GS_{(x,z)} = GS_{(x,p_x)} \cup GS_{(p_x,p_z)} \cup GS_{(p_z,z)}.$$

Since

$$GS_{(x,y)} \cap GS_{(x,z)} = GS_{(x,p_x)},$$

it follows that

$$GS_{(p_x,p_y)} \cap GS_{(p_x,p_z)} = \{p_x\}.$$

There are symmetric statements for p_y and p_z and hence $GS_{(p_y,p_x)}$, $GS_{(p_x,p_z)}$ and $GS_{(p_y,p_z)}$ overlap only at endpoints. Since, by Axiom 5, their union cannot extend to a nontrivial cyclic word, necessarily p_y, p_x and p_z are cusp points of some common cusp tree; $T[p_y, p_x]^c = T[p_x, p_z]^c = T[p_y, p_z]^c$. So we have the second possibility. □

Note that given three points x, y, and z, the geodesic spines between them split up in various ways depending on the Y configurations.

Corollary 3.11.

- *In the case where all three Y's are the single point p, then $GS_{(x,z)} = GS_{(x,p)} \cup GS_{(p,z)}$, $GS_{(x,y)} = GS_{(x,p)} \cup GS_{(p,y)}$, and $GS_{(y,z)} = GS_{(y,p)} \cup GS_{(p,z)}$.*
- *In the case where the three Y's are three distinct points, we have $GS_{(x,z)} = GS_{(x,p_x)} \cup GS_{(p_z,z)}$, and there is a cusp $[p_x, p_z]^c$ along any standard geodesic from x to z. By symmetry, $GS_{(x,y)} = GS_{(x,p_x)} \cup GS_{(p_y,y)}$, and there is a cusp $[p_x, p_y]^c$ along any standard geodesic from x to y, and $GS_{(y,z)} = GS_{(y,p_y)} \cup GS_{(p_z,z)}$, and there is a cusp $[p_y, p_z]^c$ along any standard geodesic from y to z.*
- *In the case where exactly one of the Y's is a cusp, say $Y_{(x,y,z)} = [p_x, p_z]^c$, then $GS_{(x,z)} = GS_{(x,p_x)} \cup GS_{(p_z,z)}$, $GS_{(x,y)} = GS_{(x,p_x)} \cup GS_{(p_x,y)}$, $GS_{(y,z)} = GS_{(y,p_z)} \cup GS_{(p_z,z)}$. Furthermore, note that in this case, in $GS_{(y,p_x)}$ and $GS_{(y,p_z)}$, neither p_x nor p_z can be part of a cusp pair.*

Proof. Almost all of the corollary follows immediately from the previous theorem. In the third case, we claimed that in $GS_{(y,p_x)}$ and $GS_{(y,p_z)}$, neither p_x nor p_z can be part of a cusp pair. Notice that $GS_{(y,p_x)} = X_{(x,y,z)} \cup \{p_x\}$, and that $X_{(x,y,z)}$ ends in an open segment by the definition of $Y_{(x,y,z)}$. Hence p_x is not part of a cusp pair along $GS_{(y,p_x)}$. $\square$

4. Minimal actions and fixed points

Given an action by G on an $\mathbb{R}$-tree T, we may expand T to a larger tree T' with an action by G in a rather trivial way, which we call a type 1 expansion. Choose any point $p \in T$, choose another tree S, and choose a point s on S. Now attach at every point gp for $g \in G$ a copy of S at the chosen spot. Extend the action to T', the union of T and all these copies of S, by simply acting on T as before and permuting the copies of S. T' now has an invariant subtree isomorphic to T. An action on an $\mathbb{R}$-tree is considered *minimal* if there is no invariant subtree.

If T is an order tree, there is an additional way to enlarge the tree by inserting cusps, which we call a type 2 expansion. First of all, we may select a point $p \in T$, and remove p. Now any segments in T which ended in p have become open segments. Choose one such segment and leave it open, along with all such segments which have nonempty intersection with the given segment. Now close all of the other segments by inserting new points. Of course, when two of these open segments have a non-empty intersection, they are both closed by the same new point. So we have effectively split p into many copies, each of which is a cusp point, and we have left one ray limiting on p open to be the distinguished ray of the corresponding cusp tree. Again, doing this at every point gp for $g \in G$ produces a new order tree T', as long as the orbit of p contains no accumulation points. Of course, if some group element g fixes p, then we can only add cusps in this manner if there is an open segment ending at p which is not carried by g to any pointwise distinct segments ending at p. Notice that if the tree does not really branch at p, say in the case $T = \mathbb{R}$, this move does not change T.

By combining types 1 and 2 expansions, we can expand T to a tree T' in more complicated ways. For instance, if we first choose a point p at which to perform a series of type 1 expansions, and then perform a type 2 expansion at p, we can transform even the tree $\mathbb{R}$ into a complicated object. (Naturally, we perform all expansions equivariantly.)

Notice that T is no longer necessarily a subtree of T'. But this is not too far from the case. Let T'' be the subforest of T' consisting of all of the points coming from points in T which did not get split, together with all of the cusp points coming from each point $p \in T$ which did get split. Then if we identify all of these points around a cusp into one, the result

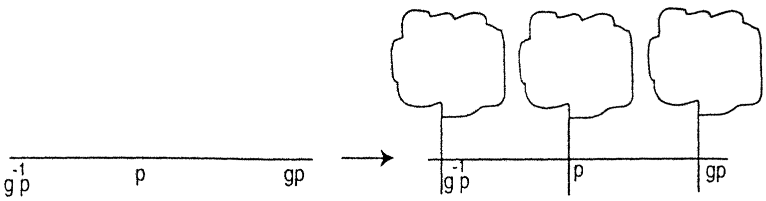

Fig. 3. Type 1 expansion.

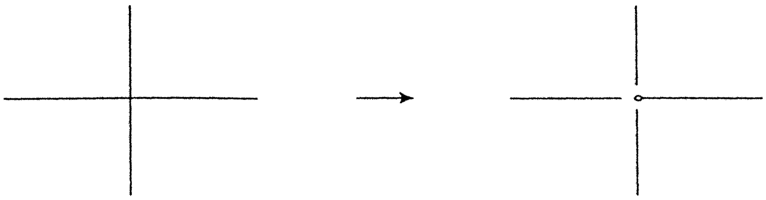

Fig. 4. Type 2 expansion.

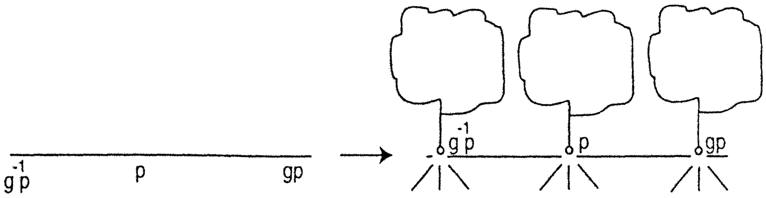

Fig. 5. A combination of types 1 and 2 expansions.

will be isomorphic to T. Notice that given two points x and y of T'', although it is not true that they can always be joined by a path within T'', it is true that $GS_{(x,y)} \in T''$. We call such a subset an *implicit subtree*.

We elaborate a bit more on the special case of an action which could be constructed by starting with an action on $\mathbb{R}$ and performing a series of the above constructions. Then there is an invariant implicit subtree consisting of a countable union of segments (possibly some trivial ones consisting of single points) σ_j, with the property that there is a cusp $[f(\sigma_j), i(\sigma_{j+1})]^c$, $\forall j$. We will call such a collection of segments an invariant *implicit* line. On the other hand, if there is an invariant implicit line, the action induces an action of G on $\mathbb{R}$. Simply identify all pairs of cusp points.

In the case of $\mathbb{R}$-trees, artificially complicated actions are ruled out by restricting to minimal actions, where a minimal action is one which has no invariant proper subtree. We clearly want to rule out a slightly more general class of actions.

In summary, we make the following definitions.

Definition 4.1. If T contains a G-invariant subset T', with the property that for any two points $x, y \in T'$, $GS_{(x,y)} \subseteq T'$, we call T' an invariant *implicit subtree*. Of course, an invariant subtree is one special case of an invariant implicit subtree.

We call T' an invariant *implicit line* if T' admits a total ordering (without a greatest or a least element) with the following property: Choose any $x, y \in T'$ and let $[x, y]$ denote the interval of T' determined by the total ordering. Then $GS_{(x,y)} = [x, y]$ and furthermore, the natural ordering on $GS_{(x,y)}$ agrees with the total order on $[x, y]$.

Definition 4.2. Let a group G act on an order tree T. The action is *minimal* if T contains no proper invariant implicit subtree.

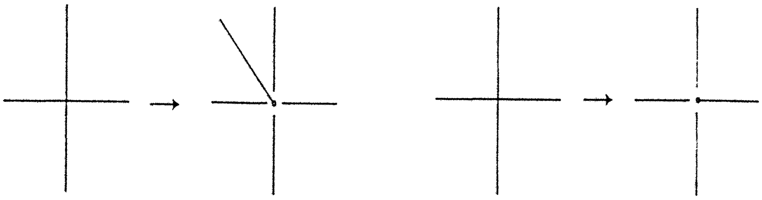

Fig. 6. Generalized fixed point.

Consider now the special case that in the above constructions, the point p at which expansions occur is fixed by some $g \in G$. In the new tree formed, g might no longer fix p. In this case, g fixes instead a corresponding cusp tree. See Fig. 6 for two examples. In the first example, let g act by rotation by $\frac{1}{2}\pi$ about p. In the second, let g permute three of the rays and fix the fourth.

More generally, it proves useful to introduce a more general notion of fixed point. We make the following definitions.

Definition 4.3. We say that $g \in G$ has a *generalized fixed point* if either $gx = x$ for some $x \in T$ or else g fixes a cusp tree.

We shall say that the group G acts *trivially* if G has a generalized fixed point. Otherwise, we say G acts *nontrivially*.

Notice that if G acts minimally and T is not a single point, then G acts nontrivially.

We close this section with a brief discussion of the notion of generalized fixed point versus fixed point. From the point of view of the algebraic arguments we use in Section 7, generalized fixed points and fixed points behave essentially the same. Although we carefully consider all of the various cases in each theorem, the core of the argument is generally not dependent on the type of fixed point or intersections of geodesic spines involved. In other words, it is really the non-metric aspect of order trees, rather than the non-Hausdorff points, which makes them behave differently from $\mathbb{R}$-trees in these algebraic arguments.

Another approach to the proofs in Section 7 would be to restrict attention to $\mathbb{R}$-order trees (i.e., to add the 8th axiom) and consider, instead of the $\mathbb{R}$-order tree itself, a related Hausdorff version of the tree (see [24] for a related construction). Consider the set of points which are cusp points of some cusp tree. Select one, say x, and consider the set of cusp trees for which x is a limit point. Blow the point x up into a star shape by attaching a segment to x for each cusp tree, so that now the other endpoints of these segments are limit points for the corresponding cusp trees. (See, for example, Proposition 3.1 and Lemma 4.1 of [18].) Do this at each point x. Now, for each cusp tree, identify the set of points which are limit points for the cusp tree. One enlarges the set of segments for the original tree in the obvious way, and the action of the group extends naturally to this new tree. The new tree will be Hausdorff, and it has genuine fixed points where the original tree had generalized fixed points.

The core of the algebraic arguments will go through on this new tree, the advantage being that geodesic spines are now genuine segments, Y's are now just ordinary intersections with no cusps, and hence there are fewer cases to consider. However, from the lamination point of view, this method is less satisfying, since cusp points in the tree reflect a structure in the lamination which is hidden when one eliminates them from the tree. For this reason we prefer, in the present article, to carefully keep track of the details in the order tree which fully reflect the structure of the underlying lamination.

5. Group actions on order trees

Many of the properties of group actions on $\mathbb{R}$-trees developed in [1,10] have analogues for order trees. In this section, we explore actions by group elements according to whether or not they have fixed points. Throughout the section, we let T be an order tree and G be a group acting by order tree isomorphisms.

First we consider the case where g has a fixed point. In the $\mathbb{R}$-tree case the fixed point set is a subtree, but since there is no metric in the order tree case, the endpoints of a segment can be fixed without the segment being pointwise fixed. To state precisely what happens, we introduce some terms.

Definition 5.1. If g has a fixed point in T, let A_g be the set of all fixed points. Furthermore, let

$$\mathit{Span}(A_g) = \bigcup_{x,y \in A_g} GS_{(x,y)}.$$

We note the following:

Proposition 5.2. *Let g have a fixed point in T. Then we have*:

(1) *$\mathit{Span}(A_g)$ is setwise fixed by g.*
(2) *If the cusp $[x,x']^c$ appears on a standard geodesic between y and z with $y,z \in A_g$, then $x,x' \in A_g$ and g fixes $T[x,x']^c$ setwise.*

Proof. From the definition of standard geodesic, it is clear that any $g \in G$ takes standard geodesics to standard geodesics. Furthermore, g takes spines of standard geodesics to spines. Therefore, since spines are unique, if $y,z \in A_g$, then g fixes $GS_{(y,z)}$ setwise. Since there are only a finite number of cusps along any spine, and they occur in a linear order, the second part of the theorem is clear. □

As was discussed in Section 4, it is necessary to expand the notion of fixed point to include the case of a fixed cusp tree. However we now show that we can always study an action with fixed cusp tree by instead studying a closely related action with fixed point.

We begin by considering adding endpoints to “bounded rays”.

Proposition 5.3. *Let P denote the set of segments of T and let $\Sigma = \{\sigma_\alpha\}$ be any subset of P. Let x_α be a collection of distinct points not in T. Set*

- $\widehat{T} = T \cup \{x_\alpha\}$.
- $\widehat{P} = P \cup \{\pm([x, f(\sigma_\alpha)) \cup \{x_\alpha\}) \mid [x, f(\sigma_\alpha)] \in P$ *and* $[x, f(\sigma_\alpha)) \cap \sigma_\alpha \neq \emptyset\} \cup \{\pm(\{x_\beta\} \cup (f(\sigma_\beta), f(\sigma_\alpha)) \cup \{x_\alpha\}) \mid [f(\sigma_\beta), f(\sigma_\alpha)] \in P$ *and* $(f(\sigma_\alpha), f(\sigma_\beta)) \cap \sigma_\alpha \neq \emptyset$ *and* $(f(\sigma_\alpha), f(\sigma_\beta)) \cap \sigma_\beta \neq \emptyset\}$.

Then $\widehat{T}$ with set of segments $\widehat{P}$ is an order tree.

Proof. Since T is an order tree, properties 1–4 follow immediately for $\widehat{T}$. Since any two segments containing x_α overlap in a set which contains a (nondegenerate) segment with endpoint x_α, property 5 also follows. If a pair of segments overlap only at an endpoint, either one, both or neither of the segments is a segment in $\widehat{P}$ but not in P. In all cases one checks that the conclusion of axiom 6 follows. □

Note that if Σ is invariant under G then the action of G on T extends to an action on $\widehat{T}$ in a natural way: define $gx_\alpha = x_\beta$, where $g\sigma_\alpha = \sigma_\beta$.

Now let $\mathcal{T}$ denote the set of all cusp trees in T. For each element of $\mathcal{T}$, $\tau_\alpha = T[x, y]^c$, choose a segment which is a representative of the distinguished ray of the cusp tree, and call it σ_α. Setting $\Sigma = \{\sigma_\alpha \mid \tau_\alpha \in \mathcal{T}\}$, we call the corresponding $\widehat{T}$ the *completion of T*. Note that although Σ might not be G-invariant, $\mathcal{T}$ is G-invariant and so once again any action of G on T extends canonically to an action of G on $\widehat{T}$: define $gx_\alpha = x_\beta$ whenever $g\tau_\alpha = \tau_\beta$. Notice that $\widehat{T}$ has cusps not present in T, namely those involving the x_α, but $\widehat{T}$ has no additional cusp trees. Furthermore, if g has a generalized fixed point in $\widehat{T}$ (say a cusp tree corresponding to $\tau_\alpha \in \mathcal{T}$), then in fact g has a fixed point in $\widehat{T}$ (namely x_α).

Proposition 5.4. *Let T be an order tree with completion $\widehat{T}$.*

(1) *G acts trivially on T if and only if G acts trivially on $\widehat{T}$.*
(2) *$g \in G$ has a generalized fixed point in T if and only if g has a* (*generalized*) *fixed point in $\widehat{T}$.*
(3) *T has a G-invariant implicit line L if and only if $\widehat{T}$ has G-invariant line L.*

Proof. If G has a generalized fixed point in $\widehat{T}$, then it has a fixed point in $\widehat{T}$. But if $Gx = x$ for some $x \in \widehat{T}$ then either $x \in T$, in which case G has a fixed point in T, or else $x = x_\alpha$ and G fixes the cusp tree τ_α, in which case G has a generalized fixed point in T. Conversely, if G acts trivially on T then either $Gx = x$ for some $x \in T \subset \widehat{T}$ or G fixes τ_α. In the latter case, G fixes x_α in $\widehat{T}$. Hence, (1) is true.

(2) follows similarly.

If T contains a G-invariant line L, then so does $\widehat{T}$. On the other hand, if $\widehat{T}$ contains a G-invariant line L then L can contain none of the points x_α (since any totally ordered set with order agreeing with the order on segments and containing x_α would necessarily contain x_α as a least or greatest element), and hence $L \subset T$. □

We now turn to group elements without generalized fixed points. Let g be an element of G without generalized fixed points. In the $\mathbb{R}$-tree case, there is an axis, a copy of $\mathbb{R}$, along which g acts by translation. This is almost true for order trees, though the notions of axis and translation are not as strong. In particular, there is always an “axis”, but it is not necessarily homeomorphic to $\mathbb{R}$. As an example, consider the foliation of the plane consisting of infinitely many Reeb components pointing alternately upwards and downwards. Consider the order tree isomorphism induced by horizontal translation. There is an “axis of translation”, but it consists of a series of cusp points. Notice that although this axis is discrete, it is still an implicit line.

Definition 5.5. If g has no generalized fixed points, let

$$A_g = \{p \in T \mid p \in GS_{(g^{-1}p,gp)}\}.$$

Theorem 5.6. *Suppose g has no generalized fixed points. Then*

(1) $g(A_g) = A_g$.

(2) *A_g is an implicit line. Furthermore, if p is any point of A_g, then*

$$A_g = \bigcup_i GS_{(g^{i-1}(p),g^i(p))}$$

and so g acts by “translation” on A_g.

Proof. Let $p \in A_g$. Then $p \in GS_{(g^{-1}(p),gp)}$. Now $g(GS_{(g^{-1}(p),gp)}) = GS_{(p,g^2p)}$, so $gp \in GS_{(p,g^2p)}$. So $g(p) \in A_g$. Hence, $g(A_g) \subset A_g$. Similarly, $g^{-1}(A_g) \subset A_g$. So $g(A_g) = A_g$ and hence we have property (1).

To show the second part, we first argue that A_g is not empty. Choose any $x \in T$. By Theorem 3.10, $Y_{(g^{-1}x,x,gx)} = p_x$ or $[p_{g^{-1}x}, p_{gx}]^c$ for some p_x or $p_{g^{-1}x}, p_{gx}$ in T. The first possibility breaks up into three subcases, so we have the following four possibilities:

(1) $Y_{(g^{-1}x,x,gx)} = Y_{(g^{-1}x,gx,x)} = Y_{(x,g^{-1}x,gx)} = p_x$.

(2) $Y_{(g^{-1}x,x,gx)} = p_x$,$Y_{(g^{-1}x,gx,x)} = p_{gx}$, $Y_{(x,g^{-1}x,gx)} = p_{g^{-1}x}$, where p_x, $p_{g^{-1}x}$, and p_{gx} are all distinct.

(3) $Y_{(g^{-1}x,x,gx)} = p_x$,$Y_{(g^{-1}x,gx,x)} = p_{gx}$, and $Y_{(x,g^{-1}x,gx)} = [p_x, p_{gx}]^c$.

(4) $Y_{(g^{-1}x,x,gx)} = [p_{g^{-1}x}, p_{gx}]^c$, in which case $Y_{(g^{-1}x,gx,x)} = p_{gx}$ and $Y_{(x,g^{-1}x,gx)} = p_{g^{-1}x}$.

Of course, we can also reverse the roles of g and g^{-1} in the third possibility to obtain a symmetric configuration.

If either of the first three cases hold, we pin down the locations of gp_x and $g^{-1}p_x$. Now

$$p_x \in GS_{(x,gx)} \Rightarrow g^{-1}p_x \in GS_{(g^{-1}x,x)}$$

and

$$p_x \in GS_{(g^{-1}x,x)} \Rightarrow gp_x \in GS_{(x,gx)}.$$

Focusing first on gp_x, either $gp_x \in GS_{(x,p_x)}$ or $gp_x \in GS_{(p_x,gx)}$. We rule out the first possibility. Suppose in fact that $gp_x \in GS_{(x,p_x)}$. Then $g^2p_x \in GS_{(gx,gp_x)}$; so p_x, g^2p_x both are in $GS_{(gx,gp_x)}$. So either $p_x \in GS_{(gp_x,g^2p_x)}$ or $g^2p_x \in GS_{(p_x,gp_x)}$. We will show that neither of these cases is possible.

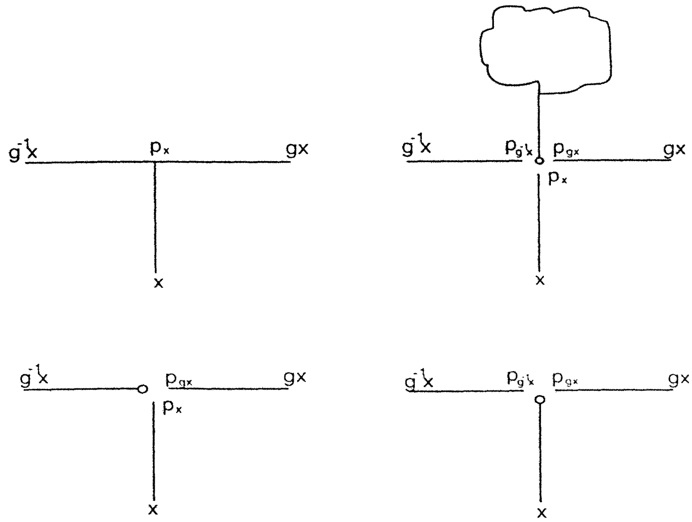

Fig. 7. Configurations for x, gx, and $g^{-1}x$.

Suppose first that $p_x \in GS_{(gp_x,g^2p_x)}$. Then

$$GS_{(p_x,gp_x)} \subseteq GS_{(gp_x,g^2p_x)} = g(GS_{(p_x,gp_x)}),$$

but g reverses the natural linear order. Now along $GS_{(gp_x,p_x)}$ are at most a finite number of cusps, and they are mapped by g onto the cusps in $GS_{(g^2p_x,gp_x)}$, which contains all of the original cusps. Hence all of the cusps in question are in $GS_{(gp_x,p_x)}$. If there are an even number of them, then between the center pair is a segment which is flipped by g, and hence g has a fixed point, a contradiction. If, however, the number is odd, then the center cusp is itself flipped by g, and g fixes the corresponding cusp tree, another contradiction.

If, on the other hand, $g^2p_x \in GS_{(p_x,gp_x)}$, then

$$GS_{(gp_x,g^2p_x)} \subset GS_{(p_x,gp_x)} = g^{-1}GS_{(gp_x,g^2p_x)},$$

and a similar argument leads to contradictions.

Hence $gp_x \notin GS_{(x,p_x)}$, in which case $gp_x \in GS_{(p_x,gx)}$. So $g^{-1}p_x \in GS_{(g^{-1}x,p_x)}$. Now that we know exactly where gp_x is, we show that in each of cases (1), (2), and (3), either p_x or p_{gx} is an element of A_g.

In case (1), since $p_x \in GS_{(g^{-1}x,gx)}$, this implies that $p_x \in GS_{(g^{-1}p_x,gp_x)}$, and hence $p_x \in A_g$.

In case (2), note that $gp_x \in GS_{(p_x,gx)}$, which implies that $gp_x \in GS_{(p_{gx},gx)}$ since p_x is not fixed by g. Since $g\,GS_{(g^{-1}x,p_x)} = GS_{(x,gp_x)}$, this in turn implies that $g(p_{g^{-1}x}) \in GS_{(p_x,gp_x)}$. However, $g(p_{g^{-1}x}) \neq p_x$, for then $g(p_x) = p_{gx}$, and g would fix the cusp tree $T[p_{g^{-1}x}, p_x]^c$ raywise. So $g(p_{g^{-1}x}) \in GS_{(p_{gx},gx)}$. Similarly, $g^{-1}(p_{gx}) \in GS_{(g^{-1}x,p_{g^{-1}x})}$. So $p_{gx} \in GS_{(g^{-1}p_{gx},gp_x)}$, and this spine ends in the cusp $[gp_{g^{-1}x}, gp_x]^c$. However, since gp_x, $g(p_{g^{-1}x})$, and $g(p_{gx})$ are all limit points of the same cusp tree, this final cusp may be replaced by $[gp_{g^{-1}x}, gp_{gx}]^c$, yielding $GS_{(g^{-1}p_{gx}, gp_{gx})}$. Hence $p_{gx} \in GS_{(g^{-1}(p_{gx}), g(p_{gx}))}$, and so $p_{gx} \in A_g$.

In case (3), note that due to the asymmetry $g(p_x) \neq p_{gx}$, and $g(p_x) \in GS_{(p_{gx}, gx)}$. Now p_{gx} forms a cusp with p_x, so $g(p_{gx})$ forms a cusp with $g(p_x)$, and $g^{-1}(p_{gx})$ forms a cusp with $g^{-1}(p_x)$. Now $p_{gx} \in GS_{(g^{-1}p_x, gp_x)}$, and cannot equal either endpoint. One observes that the right hand end of this geodesic spine ends in part of the ray of $T([gp_x, gp_{gx}]^c)$, closed off by gp_x; so we may replace the endpoint by gp_{gx} and still have a geodesic spine. Similarly, since the left-hand end of the geodesic spine ends in $[g^{-1}p_x, g^{-1}p_{gx}]^c$, we may delete the endpoint $g^{-1}p_x$, and then we see that $p_{gx} \in GS_{(g^{-1}(p_{gx}), g(p_{gx}))}$. Hence once again, $p_{gx} \in A_g$.

The fourth case is very similar to all the others. Imitating the arguments at the beginning, it is not hard to see that $gp_{g^{-1}x} \in GS_{(p_{gx}, gx)}$ and that $g^{-1}p_{gx} \in GS_{(g^{-1}x, p_{g^{-1}x})}$, and then keeping careful track of where the various cusp trees are in the configuration, as in case (3), one sees that both $p_{gx} \in GS_{(g^{-1}(p_{gx}), g(p_{gx}))}$ and $p_{g^{-1}x} \in GS_{(g^{-1}(p_{g^{-1}x}), g(p_{g^{-1}x}))}$, and hence both p_{gx} and $p_{g^{-1}x} \in A_g$.

Now choose $p \in A_g$, and let

$$L = \bigcup_i GS_{(g^{i-1}(p), g^i(p))}.$$

By the definition of A_g, these geodesic spines intersect only at the endpoints. Furthermore, it is clear that $L \subset A_g$. On the other hand, suppose that $x \in A_g$. We must show that $x \in L$. Now $x \in GS_{(g^{-1}x, gx)}$. Consider $L \cap GS_{(x,p)}$, which certainly contains p. This intersection has one of the following forms:

(1) $GS_{(p,m)}$ for some $m \in GS_{(x,p)}$.
(2) $GS_{(p,m')} \setminus \{m'\} = GS_{(p,m)} \setminus \{m\}$ for some $m' \in L$ and $m \in GS_{(x,p)}$.
(3) $GS_{(p,m)} \setminus \{m\}$ for some $m \in GS_{(x,p)}$, and the intersection has no limit point on L at the end away from p.

Note that case (3) cannot occur, for g cannot fix m, as g has no fixed points. But then gm is another limit point for the open end of $GS_{(p,m)} \setminus \{m\}$, and m and gm form a cusp, whose cusp tree (which is one end of L) is fixed by g. But g had no generalized fixed points, so this is impossible.

Note that if q is another point of L such that neither one of $GS_{(x,p)}$ and $GS_{(x,q)}$ contains the other, then the configuration above, with q instead of p, may have a different form. However, once again we have four possibilities for the way things are configured (see Fig. 8), based on the various Y's for the points x, p, and q, just as in the previous section of this proof.

Since g acts on L, the g-images of these configurations will look exactly the same. See Fig. 9.

From the pictures one can read off the geodesic spine from $g^{-1}x$ to gx in each case, and it is clear that the only way that x can be on this geodesic spine is if we have the first case, and $x = m$, which shows that in fact $x \in L$.

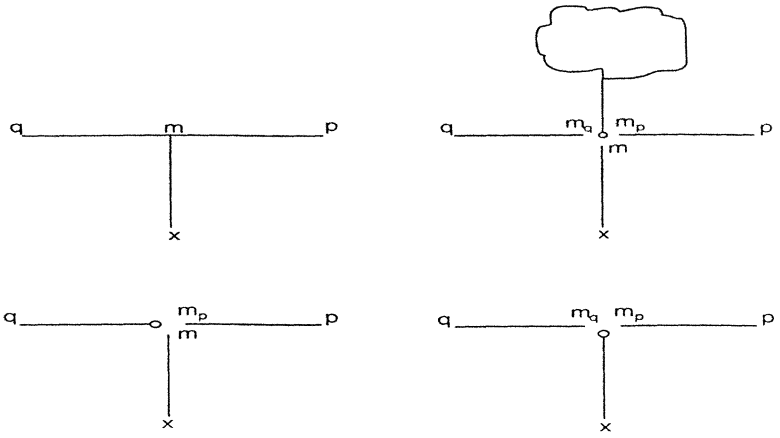

Fig. 8. Configurations of x and L.

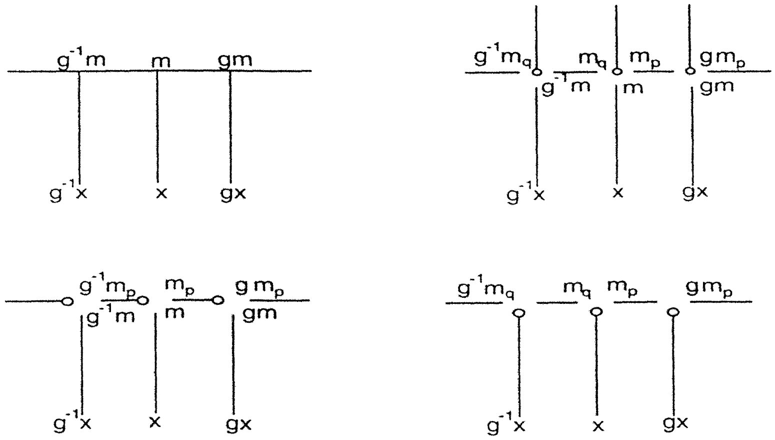

Fig. 9. Configurations of x, gx, $g^{-1}x$, and L.

Finally, we note that L has a natural linear ordering since $GS_{(g^{i-1}p,g^i p)}$ is linearly ordered for all i and $g^i p \neq p$ for any i. To see the latter, note that $g^i p = p$ for some i would imply either the existence of a nontrivial cycle in T or else that $\{p, gp, \ldots, g^{i-1}p\}$ are all cusp points of a common cusp tree and hence that g has a generalized fixed point. □

Note. In the above proof, Fig. 9 reveals how g acts on a general $x \in T$. In particular, it is easy to see the following corollary.

Corollary 5.7. *If g has no generalized fixed points, then no nonzero power of g has generalized fixed points.*

We conclude this section with a routine lemma about the G-action on the sets A_g.

Lemma 5.8. $h(A_g) = A_{hgh^{-1}}$, $\forall g \in G$.

6. Preliminary lemmas

We next prove some results useful in the proof of Theorem 7.2 and perhaps of independent interest.

Lemma 6.1. *Let G act on the order tree T. Suppose $g \in G$, and $s, t, x \in T$ such that $gs = s$, $g^\alpha x = x$ and $x, gx \in GS_{(s,t)}$. Then $gx = x$.*

Proof. Without loss of generality, $\alpha \geqslant 1$. Suppose $gx \in GS_{(s,x)}$. Then

$$g\, GS_{(s,x)} = GS_{(s,gx)} \subset GS_{(s,x)}\,.$$

So, by induction,

$$GS_{(s,x)} = g^\alpha\, GS_{(s,x)} \subset \cdots \subset g\, GS_{(s,x)} \subset GS_{(s,x)}$$

and hence $g\, GS_{(s,x)} = GS_{(s,x)}$. So $gx = x$.

Otherwise, $g^{-1}x \in GS_{(s,x)}$ and a similar argument reveals that $g^{-1}x = x$.

Hence, in either case, $gx = x$. □

Corollary 6.2. *Let G act on an order tree T. Suppose $g \in G$ and $x, y \in T$ such that $gy = y$, $g^\alpha x = x$.*

- *If $Y_{(x,y,gx)} = p$ for some $p \in T$, then $gp = p$. Furthermore, in this case, either $Y_{(y,x,gx)} = Y_{(x,gx,y)} = p$, or $Y_{(y,x,gx)} = p_x$, $Y_{(x,gx,y)} = p_{gx}$, and $p = p_y$, p_x, and $p_{gx} = g(p_x)$ are all distinct.*
- *If $Y_{(x,y,gx)} = [p_x, p_{gx}]^c$ for some $p_x, p_{gx} \in T$ then $gp_x = p_{gx}$.*

Proof. Note that g^α fixes both x and gx.

If $Y_{(x,y,gx)} = p$, then $GS_{(x,y)} \cap GS_{(gx,y)} = GS_{(y,p)}$. So $p, gp \in GS_{(gx,y)}$. Also, $g^\alpha p = p$ since

$$GS_{(y,g^\alpha p)} = g^\alpha \big(GS_{(x,y)} \cap GS_{(gx,y)}\big) = GS_{(x,y)} \cap GS_{(gx,y)} = GS_{(y,p)}\,.$$

Therefore, by Lemma 6.1, $gp = p$. If $Y_{(y,x,gx)}$ were a cusp $[p, p_{gx}]^c$,

$$GS_{(gx,p)} = GS_{(gx,p_{gx})} \cup [p_{gx}, p]^c,$$

whereas $GS_{(x,p)} = X_{(y,x,gx)} \cup \{p\}$, where $X_{(y,x,gx)}$ ends in an open segment, so that $GS_{(x,p)}$ does not involve a cusp at p (see Corollary 3.11). However, $g\, GS_{(x,p)} = GS_{(gx,p)}$ so $Y_{(gx,x,y)}$ cannot be a cusp. Similarly, $Y_{(x,gx,y)}$ cannot be a cusp either, so case (1) or case (2) of Theorem 3.10 holds, and in case (2), since $g\, GS_{(x,p)} = GS_{(gx,p)}$ it is clear that $g(p_x) = p_{gx}$.

Otherwise, $Y_{(x,y,gx)} = [p_x, p_{gx}]^c$ for some $p_x, p_{gx} \in T$, and we have case (3) of Theorem 3.10. Since $g^\alpha\, GS_{(x,gx)} = GS_{(x,gx)}$, a cusp count reveals that $g^\alpha p_x = p_x$. Now

$gp_x \in GS_{(gx,y)}$. If $gp_x \in GS_{(p_x,y)}$, then by Lemma 6.1 $gp_x = p_x$, a contradiction. So $gp_x \in GS_{(p_{gx},gx)}$. If $gp \neq p_{gx}$, then $g^{-1}(p_{gx}) \in GS_{(p_{gx},y)}$ and again Lemma 6.1 yields a contradiction. Therefore, $gp_x = p_{gx}$. □

7. Non-Haken Seifert fibered spaces

Non-Haken Seifert fibered spaces (i.e., those which do not contain an incompressible surface) form a subset of the exceptional Seifert fibered spaces (i.e., those Seifert fibered spaces with 3 exceptional fibres and base space S^2). When M is an exceptional Seifert fibered space it is possible to completely describe all minimal order tree actions of $\pi_1(M)$.

It is known which Seifert fibered spaces contain foliations transverse to the Seifert fibering and which do not [11,31]. In [33] we use this fact to completely describe all order tree actions of $\pi_1(M)$ on $\mathbb{R}$ by showing:

Theorem 7.1. *If M is an exceptional Seifert-fibered space, then $\pi_1(M)$ acts nontrivially on the order tree $\mathbb{R}$ via α if and only if M contains a foliation transverse to the Seifert fibering and with space of leaves $\mathbb{R}$ on which the associated $\pi_1(M)$ action is conjugate to α.*

We now complete the picture by demonstrating that there are no nontrivial minimal actions on any order tree other than $\mathbb{R}$.

Theorem 7.2. *Let G be the fundamental group of an exceptional Seifert fibered space. If G acts nontrivially and minimally on an order tree T, then $T = \mathbb{R}$. In fact, if the action is nontrivial but not minimal, then there is an invariant implicit line.*

Proof. When M is an exceptional Seifert-fibered space,

$$\pi_1(M) \cong \langle a_1, a_2, a_3, z \mid {a_1}^{\alpha_1} = z^{\beta_1},\ {a_2}^{\alpha_2} = z^{\beta_2},\ {a_3}^{\alpha_3} = z^{\beta_3},\ a_1a_2a_3 = z^b,\ z \text{ is central}\rangle.$$

Since ${a_i}^{\alpha_i} = z^{\beta_i}$, $1 \leqslant i \leqslant 3$, it follows from Corollary 5.7 that z has a generalized fixed point if and only if each a_i has a generalized fixed point, $1 \leqslant i \leqslant 3$.

If z has no generalized fixed points, then by Lemma 5.8, z acts on A_z and $a_i(A_z) = A_{a_i z a_i^{-1}} = A_z$ since z is central. Hence G acts on A_z, and so the action either has a proper invariant implicit line, or $T = A_z = \mathbb{R}$.

So suppose z has a generalized fixed point. Then all four generators have fixed points, and it turns out that the action is trivial. We have organized the proof of this fact into two lemmas below. We first prove that if z and any other generator have generalized fixed points, then the pair has a common generalized fixed point. We then complete the proof by showing that if each pair z, a_i has a common generalized fixed point, then the entire group has a common generalized fixed point and so the action was trivial, a contradiction. □

Lemma 7.3. *Let G be any group with elements z, a satisfying $z^\beta = a^\alpha$ and with z central in G. Let T be an order tree and suppose G acts on T so that a and z both have generalized fixed points. Then a and z have a common generalized fixed point.*

Proof. Consider first the special case that z and a both have fixed points; $as = s, zt = t$ say, for some $s, t \in T$. By Corollary 6.2, at least one of the following is true:

- $Y_{(s,t,zs)} = p$ for some $p \in T$, $zp = p$.
- $Y_{(t,s,at)} = q$ for some $q \in T$, $aq = q$.
- $Y_{(s,t,zs)} = [m, zm]^c$ and $Y_{(t,s,at)} = [n, an]^c$ for some $m, n \in T$.

Suppose first that $Y_{(x,y,zx)} = p$ for some $p \in T, zp = p$. If the first possibility of case (1) of Corollary 6.2 holds true, then $p \in GS_{(s,zs)}$. Since $a\,GS_{(s,zs)} = GS_{(s,zs)}$, $ap \in GS_{(s,zs)}$ also. Therefore, since $a^\alpha(p) = z^\beta(p) = p$, Lemma 6.1 guarantees that $ap = p$. So z and a both fix p. Otherwise, the second possibility in case (1) of Corollary 6.2 holds. Let $p_s = Y_{(t,s,zs)}$. By Corollary 6.2, $Y_{(s,zs,t)} = p_{zs} = zp_s$. Since $a\,GS_{(s,zs)} = GS_{(s,zs)}$, cusp counting reveals that $a[p, p_s]^c = [p, p_s]^c$. So $T[p, p_s]^c = T[p_s, p_{zs}]^c$ is fixed by both a and z. Hence, in either case, a and z share a generalized fixed point.

Similarly, if $Y_{(t,s,at)} = q$ for some $q \in T, aq = q$, then a and z share a generalized fixed point.

So suppose the third possibility holds true. Since $a\,GS_{(s,zs)} = GS_{(s,zs)}$, a cusp count reveals that a fixes both m and zm. Similarly, z fixes both n and an. Furthermore, since $n, m \in GS_{(s,t)}$, either $m \in GS_{(t,n)}$ (in which case $n \in GS_{(s,m)}$) or $m \in GS_{(n,s)} \setminus \{n\}$ (in which case $n \in GS_{(t,m)} \setminus \{m\}$). But since a fixes m and zm (and hence $T[m, zm]^c$), and $t \in T[m, zm]^c$, $at \in T[m, zm]^c$. So $GS_{(t,at)} \subset T[m, zm]^c$ and in particular, $m \notin GS_{(t,at)} = GS_{(t,n)} \cup GS_{(an,at)}$. So necessarily, $m \in GS_{(n,s)} \setminus \{n\}$. See Fig. 10.

Now note that $GS_{(m,n)}, GS_{(zm,n)}, GS_{(m,an)}, GS_{(zm,an)}$ each differ only at the endpoints. So if we let L be the intersection of the four, L is invariant under the action of both a and z and both a and z fix the ends of L. Hence a and z have two common generalized fixed points.

Hence, when z and a both have fixed points, a and z share a common generalized fixed point.

Consider now the case where we know only that a and z have generalized fixed points. Let $\widehat{T}$ be the completion of T. In the associated G action, a and z act on $\widehat{T}$ and have fixed

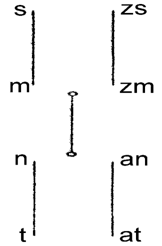

Fig. 10. Case (3).

points. Hence, by the special case, a and z share a common generalized fixed point in $\widehat{T}$. It follows that a and z share a common generalized fixed point in T by Proposition 5.4. $\square$

Lemma 7.4. *Let G be as in Theorem 7.2. Suppose that each pair z, a_i, $1 \leqslant i \leqslant 3$, has a common generalized fixed point. Then the G action is trivial.*

Proof. Consider first the special case that each pair z, a_i, $1 \leqslant i \leqslant 3$, has a fixed point.

Let r be the point fixed by both z and a_3. Note that since $a_1a_2a_3 = z^n$, $a_1a_2(r) = r$. Also, since ${a_1}^{\alpha_1}(r) = z^{\beta_1}(r) = r$, $a_2(r) = a_1^{\alpha_1 - 1}(r)$. Now choose points s_1, fixed by both a_1 and z, and s_2, fixed by both a_2 and z. By Corollary 6.2, either

- $GS_{(r,a_1r)} = GS_{(r,m_1)} \cup GS_{(m_1,a_1r)}$, where the two spines on the right hand side intersect exactly at $m_1 = a_1m_1$;
- $GS_{(r,a_1r)} = GS_{(r,m_1)} \cup GS_{(a_1m_1,a_1r)}$, where the two spines on the right hand side are disjoint and $\{m_1, a_1m_1\}$ is a cusp pair;

and either

- $GS_{(r,a_2r)} = GS_{(r,m_2)} \cup GS_{(m_2,a_2r)}$, where the two spines on the right hand side intersect exactly at $m_2 = a_2m_2$;
- $GS_{(r,a_2r)} = GS_{(r,m_2)} \cup GS_{(a_2m_2,a_2r)}$, where the two spines on the right hand side are disjoint and $\{m_2, a_2m_2\}$ is a cusp pair.

Suppose first that $GS_{(r,a_1r)} = GS_{(r,m_1)} \cup GS_{(m_1,a_1r)}$, and $GS_{(r,a_2r)} = GS_{(r,m_2)} \cup GS_{(m_2,a_2r)}$, m_1, m_2 as above. Then by examining

$$GS_{(m_1,r)}, a_1\, GS_{(m_1,r)}, \ldots, a_1^{\alpha_1 - 1}\, GS_{(m_1,r)} = GS_{(m_1,a_2r)},$$

we see that $GS_{(r,a_2r)} = GS_{(r,m_1)} \cup GS_{(m_1,a_2r)}$. See Fig. 11 for intuition, though one may have $a_1^n(r) = r$ for $n < \alpha_1$, which will not affect the argument. So $m_2 \in GS_{(r,m_1)}$ or $m_2 \in GS_{(m_1,a_2r)}$. In the first case, $a_1m_2 \in GS_{(m_1,a_1r)}$. So $m_2, a_1m_2 \in GS_{(r,a_1r)}$ and noting that $a_1m_2 = a_1a_2m_2$, apply Lemma 6.1 with $g = a_1a_2$ to obtain $a_1m_2 = m_2$. In the second case, $m_1 \in GS_{(r,m_2)}$ and a symmetric argument reveals that $a_2m_1 = m_1$. In either case, there is a point fixed by z, a_1 and a_2, and hence G.

Suppose next that $GS_{(r,a_1r)} = GS_{(r,m_1)} \cup GS_{(a_1m_1,a_1r)}$, where the two spines on the right hand side are disjoint and $\{m_1, a_1m_1\}$ is a cusp pair. Then $\{a_1^{\alpha_1}m_1 = m_1, a_1m_1, \ldots, a_1^{\alpha_1 - 1}m_1\}$ are limit cusp points of $T[m_1, a_1m_1]^c$ and

$$GS_{(r,m_2)} \cup GS_{(a_2m_2,a_2r)} = GS_{(r,a_2r)} = GS_{(r,{a_1}^{\alpha_1 - 1}r)} = GS_{(r,m_1)} \cup a_1^{\alpha_1 - 1}\, GS_{(m_1,r)}\,.$$

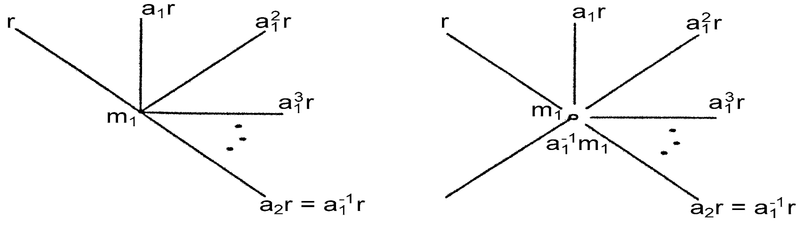

Fig. 11.

Counting cusps therefore reveals that $\{m_2, a_2m_2\}$ is necessarily a cusp pair and $m_1 = m_2, a_1^{\alpha_1-1}m_1 = a_2m_2$. So $T[m_1, a_1m_1]^c$ is fixed by a_1, a_2 and z, and hence by G.

Similarly, if $GS_{(r,a_2r)} = GS_{(r,m_2)} \cup GS_{(a_2m_2,a_2r)}$, where the two spines on the right hand side are disjoint and $\{m_2, a_2m_2\}$ is a cusp pair, then $T[m_2, a_2m_2]^c$ is fixed by G.

Thus, if each pair z, a_i, $1 \leqslant i \leqslant 3$, has a fixed point, G acts trivially.

Finally, consider the general case: each pair $z, a_i, 1 \leqslant i \leqslant 3$, has a generalized fixed point. Let $\widehat{T}$ be the completion of T. Then G acts on $\widehat{T}$ so that each pair z, a_i, $1 \leqslant i \leqslant 3$, has a common fixed point. Hence the proof of the special case guarantees that z, a_1, a_2, and a_3 share a common generalized fixed point in $\widehat{T}$. But then Proposition 5.4 guarantees that z, a_1, a_2, and a_3 share a common generalized fixed point in T. $\square$

Corollary 7.5. *Let M be an exceptional Seifert fibered space. Then every nontrivial, minimal action of $\pi_1(M)$ on an $\mathbb{R}$-order tree T is conjugate to the standard action of $\pi_1(M)$ on the space of leaves of a transverse foliation in M.*

Proof. Suppose $\pi_1(M)$ acts nontrivially and minimally on an $\mathbb{R}$-order tree T. By Theorem 7.2, $T = \mathbb{R}$. So by Theorem 7.1, M contains a foliation transverse to the Seifert fibres and the given action is conjugate to the standard one. $\square$

This translates to a new proof of a result due to Brittenham [5] and Claus [8] demonstrating the existence of nonlaminar 3-manifolds with infinite fundamental group.

Corollary 7.6. *Let M be an exceptional Seifert fibered space. Then M contains an essential lamination* (*if and*) *only if M contains a foliation transverse to the Seifert fibers. In particular, there exist nonlaminar* 3*-manifolds with infinite fundamental group.*

Proof. In Section 8, we show that if M contains an essential lamination, then $\pi_1(M)$ acts nontrivially and minimally on an $\mathbb{R}$-order tree. Therefore, by Corollary 7.5, M contains a foliation transverse to the Seifert fibering.

Furthermore, there are exceptional Seifert fibered spaces with infinite fundamental group but containing no transverse foliation [11,23]. $\square$

8. Essential laminations

As noted in Section 2, order trees were introduced by Gabai and Oertel [20] to describe the structure of the space of leaves $T(\lambda)$ of an essential lamination λ in a 3-manifold M. Recall that the action of $\pi_1(M)$ on $\widetilde{M}$ by covering translations induces an action of $\pi_1(M)$ on $T(\lambda)$, which we call the *standard action* of $\pi_1(M)$ on $T(\lambda)$. We begin by noting that this action is almost always nontrivial.

Proposition 8.1. *Let λ be an essential lamination in a compact* 3*-manifold M. Suppose that no leaf of λ is isotopic to or double covers a component of ∂M. Then the standard action of $\pi_1(M)$ on $T(\lambda)$ is nontrivial.*

Proof. If the standard action of $\pi_1(M)$ on $T(\lambda)$ has global fixed cusp tree, consider the sublamination $\lambda_0 \subset \lambda$ obtained by removing all leaves corresponding to points in the fixed cusp tree. Then the point in $T(\lambda_0)$ corresponding to the complementary region created in $\widetilde{M}$ is fixed by the action of $\pi_1(M)$, and λ_0 satisfies the conditions of the proposition. So it suffices to rule out global fixed points.

If M is nonorientable, we consider instead the action of $\pi_1(\widehat{M})$, where $\widehat{M}$ is the orientable double cover of M. Certainly, if $\pi_1(\widehat{M})$ acts nontrivially on $T(\lambda)$, $\pi_1(M)$ does also. So we may assume M is orientable.

Now let x be any point in T. Then $x = [\widetilde{L}]$, where $\widetilde{L}$ is the lift of some leaf L in λ. (If x is the equivalence class of the closure of a complementary region X of λ, let L be a boundary leaf of X.)

If γ is a simple closed curve efficient with respect to λ, then by Lemma 2.7 of [20], any lift $\tilde{\gamma}$ of γ is efficient with respect to $\tilde{\lambda}$. If in addition γ has nonempty intersection with L, then $\tilde{\gamma}$ describes a copy of $\mathbb{R}$ in $T(\lambda)$ and $[\gamma]$ acts on each such copy of $\mathbb{R}$ via translation. Since x is on one such copy, $[\gamma]$ does not fix x. Hence, it suffices to demonstrate the existence of a simple closed curve γ efficient with respect to λ and with nonempty intersection with L.

When L is noncompact, a standard argument yields γ. For L can be isotoped to lie in an I-fibered regular neighbourhood of an essential branched surface B so that it intersects each I-fibre transversely. Since L is noncompact, it meets some I-fibre infinitely often. In particular, if L is orientable, it intersects the fibre I at points t_1 and t_2 with the same orientation. An arc in L from t_1 to t_2 together with the subarc of the I-fibre from t_1 to t_2 can be perturbed to a transverse efficient loop γ. When L is nonorientable, we repeat this argument for $S = \partial N(L)$, where the regular neighbourhood $N(L)$, a twisted I-bundle, is chosen small enough to lie within B and so that its I-fibering is inherited from the I-fibering of an I-fibered regular neighbourhood of B.

Now suppose that L is compact and orientable. In this case, there might be no such γ. So we instead locate a simple closed curve efficient with respect to the sublamination $L \subset \lambda$ and having nonempty intersection with L. If L is nonseparating, the existence of such a γ is immediate. When L is separating, $M = M_1 \cup_L M_2$, where by assumption, neither M_1 nor M_2 is homeomorphic to $L \times I$ or to a twisted I-bundle over a boundary component of M. So there are arcs α_1 in M_1 and α_2 in M_2 with boundary on L whose union yields the desired γ. In either case, let $\tilde{\gamma}$ be the lift of γ which contains x. Now $\tilde{\gamma}$ is efficient with respect to L (although not necessarily with respect to λ). So $\bigcup_i GS_{([\gamma]^i x, [\gamma]^{i+1} x)}$ is an implicit line upon which $[\gamma]$ acts by translation. Finally, if L is compact but nonorientable, repeat this argument for $\partial N(L)$.

In all cases, $[\gamma]x \neq x$. $\square$

When λ is measured, then $T(\lambda)$ is in fact an $\mathbb{R}$-tree [28,29] and the standard action is minimal [21]. In the more general setting of essential laminations, however, the standard action might be nonminimal. Consider, for example, any one of the exceptional Seifert fibered spaces which also fibre over S^1 with surface fibre a torus. As noted by Brittenham [7], these spaces contain essential laminations λ obtained by taking one or more copies of the torus fibre and foliating the I-bundles complementary to this surface by essential (open) annuli which cannot be made transverse to the Seifert fibers. $T(\lambda)$ has a $\pi_1(M)$-invariant implicit subtree, obtained by deleting those points corresponding to lifts of the annuli leaves, and hence the standard action is not minimal.

However, after possibly passing to a sublamination λ, the standard action is minimal. We first note that a G-invariant implicit subtree of $T(\lambda)$ is exactly the space of leaves of a sublamination of λ.

Proposition 8.2. *Let $\lambda \subset M$ be as in the statement of Proposition* 8.1. *Let q denote the quotient map $\widetilde{M} \to T(\lambda) = \widetilde{M}/\sim$, and let p denote the quotient map $\widetilde{M} \to M$. Let T_0 be a $\pi_1(M)$-invariant implicit subtree of $T(\lambda)$. Set $\tilde{\lambda_0} = \tilde{\lambda} \cap q^{-1}(T_0)$ and $\lambda_0 = p(\tilde{\lambda_0})$. Then λ_0 is closed and hence is a sublamination of λ.*

Proof. Suppose, by way of contradiction, that λ_0 is not closed. Then there is a leaf L in $\overline{\lambda_0} \setminus \lambda_0$. Let $\widetilde{L}$ be a lift of L to $\widetilde{M}$. Note that $[\widetilde{L}] \notin T_0$. So $\tilde{\lambda_0}$ is contained in one of the half-spaces $\mathbb{R}^3 \setminus \widetilde{L}$. For if not, we can choose one leaf of $\tilde{\lambda_0}$ in each half space, and hence any path between the q-images of these two leaves must contain $[\widetilde{L}]$. But then the geodesic spine between these two points contains $[\widetilde{L}]$. But since T_0 is an invariant implicit subtree this entire geodesic spine is contained in T_0, which is a contradiction. Therefore, all translates $G\widetilde{L}$ are also contained in the closure of this half-space (since they also lie in $\overline{\tilde{\lambda_0}}$). But this is impossible. For, as in the proof of Proposition 8.1, let γ be any simple closed curve having nonempty efficient intersection with L. Choose a lift $\tilde{\gamma}$ of γ which has nonempty intersection with $\widetilde{L}$. Then $[\gamma]\widetilde{L}$ and $[\gamma]^{-1}\widetilde{L}$ lie on opposite sides of $\widetilde{L}$.

Hence, λ_0 is closed. □

Note that if T_0 is a proper implicit subtree of T, then λ_0 is a proper sublamination of λ. Hence, if λ contains no proper sublamination (i.e., is *minimal*), the standard action is minimal.

Corollary 8.3. *If λ is a minimal essential lamination, then the standard action is minimal.*

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